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A RESULT ON THE MEASURE OF PSEUDO-RANDOMNESS OF GRAPHS THROUGH AN EIGENVALUE

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ABSTRACT

We study a survey paper about pseudo-random graphs by Krivelevitch and Sudakov [1], there they have considered (n, d, λ)-graphs and we too. In this paper present some of the theorems and lemmas from it and we answer for their question how small can be second adjacency eigenvalue of graph that measure the pseudo randomness of graphs. We extended it to the second normalized Laplacian eigenvalue to measure the pseudo randomness of graph.

KEYWORDS: Eigenvalues, Random Graphs, Pseudo Random Graphs

1.1. INTRODUCTION

Last few years brought many new and striking results on pseudo-randomness by various researchers. There are two clear trends in recent research on pseudo-random graphs. The first is to apply very diverse methods from different fields (algebraic, linear algebraic, combinatorial, probabilistic etc.) to construct and study pseudo-random graphs. The second and equally encouraging is to find applications, in many cases quite surprising, of pseudorandom graphs to problems in Graph Theory, Computer Science and other disciplines. This mutually enriching interplay has greatly contributed to significant progress in research on pseudo-randomness achieved lately.

Pseudo-random graphs are graphs which behave like random graphs. Random graphs have proven to be one of the most important and fruitful concepts in modern Combinatorics and Theoretical Computer Science. Pseudo random graphs are modeled after truly random graphs, and therefore mastering the edge distribution in random graphs can provide the most useful insight on what can be expected from pseudo-random graphs. For random graph models we refer a survey paper uniform random regular graph by N C Wormald [2] and basic results in graph spectra we use the following books [3,4 & 8].

In this paper we study the survey paper about pseudo-random graphs by Krivelevitch and Sudakov [1]. We list out some of the theorems and lemmas from it and we answer the question how small can be second adjacency eigenvalue of graph that measure the pseudo randomness of graphs. We extended it to the second normalized Laplacian eigenvalue to measure the pseudo randomness of graph. The paper is organized as follows, 1.1 Brief introduction of random graphs and Pseudo random graphs. 1.2 Basic facts about random graphs and pseudo random graphs. 1.3 Contains main results with examples. At the end we conclude in section 1.4.

1.2 BASICS OF RANDOM GRAPH AND PSEUDO-RANDOM GRAPHS

1.2.1 Definition: Random Graph: A random graph $G(n, p)$ is a probability space of all labeled graphs on n vertices {1, 2, . . .n} where {i, j}is an edge of $G(n, p)$ with probability $p = p(n)$, independently of any other edges, for $1 \leq i < j \leq n$. Equivalently, the probability of a graph G = (V, E) with n vertices and e edges in G(n p) is $p^e(1-p)^{\binom{n}{2}-e}$. We observe that for $p = 1/2$ the probability of every graph is the same and for $p > 1/2$ the probability of a graph G_1 with more edges than another graph G_2 is higher. (And the probability of G_1 is smaller than the probability of G_2 if $p < 1/2$.) Almost all properties hold for all $G \in G(n, p)$ has property p tends to one as n tends to infinity. The following definition and theorem are concerned about the edge distribution of random graphs.

1.2.2 Definition: (e (U W)) The number of edges between two disjoint subsets U, W of vertices is denoted by e (U W). More generally, we define $e(U\ W) = \sum_{u \in U} \sum_{v \in V: u \sim v} 1$.

Note that if we have an edge e with both endpoints in U∩W then this edge is counted twice in $e(U\ W)$.

1.2.3 Theorem: (Random Graph: Edge Distribution ([1], p.4)): Let $p = p(n) \leq 0.9$. Then for every two (not necessarily disjoint) subsets U, W of vertices $\left|e(U\ W) - p|U||W|\right| = O\left(\sqrt{|U||W|np}\right)$

For almost all $G \in G(n, p)$. The expected degree of a vertex v in a random graph $G(n, p)$ is the same for all v. We consider in the following an extended random graph model for a general degree distribution.

1.2.4 Definition (Random Graph with Given Degree Distribution): Given $w = w_1, w_2, \ldots w_n$ a sequence. A random graph with given degree distribution $G(w)$ is a probability space of all labeled graphs on n vertices {1, 2,.,., n} where {i, j} is an edge of $G(w)$ with probability $w_i w_j \rho$, independently of any other edges, for $1 \leq i < j \leq n$. Where ρ plays the role of a normalization factor, i.e. $\rho = (\sum_{i=1}^{n} w_i)^{-1}$.

The expected degree of the vertex i is w_i. The above definition of a random graph with given degree distribution comes from Chung and Lu [5]. There are some other definitions for random graphs with given degrees (interested reader can see Watts, D. J., and et. all. [6]). For basics facts about Pseudo-random graphs we can refer [1] and therein references. Next we define the concept of pseudo-random graphs by means of the eigenvalues of a graph. Then we can make some statements about the edge-distribution of G. In the following we have

1.2.5 Definition (Second Adjacency Eigenvalue): The second adjacency eigenvalue λ (A (G)) is defined as

$$\lambda(A(G)) = \max\{-\lambda_1(A(G)),\ \lambda_{n-1}(A(G))\}$$

$$= \max_{i \neq n} |\lambda_i(A(G))|$$

This will be our link to spectral graph theory.

1.2.6 Definition ((n, d, λ)-Graph: A (n, d, λ)-Graph): is a d-regular graph on n vertices with second adjacency eigenvalue at most λ .

1.2.7 Proposition ([3], Chapter 9): Let G = (V, E) be a (n, d, λ)-graph. Then for every subset B of V we have

$$\sum_{v\in V}\left(|N_B(v)| - \frac{|B|d}{n}\right)^2 \le \lambda \frac{|B|(n-|B|)}{n}$$

Where $N_B(v)$ denotes the set of all neighbors in $B$of v.

1.2.8 Theorem ((n,d,λ)-Graph: Edge Distribution ([3], Chapter 9)): Let G be a (n,d,λ)-graph Then for every two subsets B, C of vertices, we have $\left|e(B\ C) - \frac{d|B||C|}{n}\right| \le \lambda\sqrt{|B||C|}$

If for a graph G the second adjacency eigenvalue $\lambda\left(A(G)\right)$ is small then the edge distribution of G is almost the same as for the random graph $G(n, \frac{d}{n})$ by the above theorem graph G is very pseudo-random.

There is a generalization of Theorem 1.2.8 with an improved error term.

1.2.9 Theorem ((n,d,λ)-Graph: Edge Distribution ([4] p.11)): Let G be a (n,d,λ)-graph Then for every two subsets $,W \subseteq V$,

$$\left|e(U,W) - \frac{d|U||W|}{n}\right| \le \lambda\sqrt{|U|\,|W|\left(1-\frac{|U|}{n}\right)\left(1-\frac{|W|}{n}\right)}$$

So the second adjacency eigenvalue of a graph G is a measure of the pseudo randomness of G. A graph has a very small second adjacency eigenvalue than it is very pseudo-random. In the next following theorem we answer the question how small can λ be.

1.3 MAIN RESULT I

1.3.1 Theorem: Let G be a d-regular graph on n vertices with adjacency eigenvalues $\lambda_1 \nleq \lambda_2 \le \lambda_3 \le \ldots\lambda_n$.

Then $\max_{i\neq n} |\lambda_i| \ge \sqrt{\frac{d(n-d)}{n-1}}$, In particular if $d \le 0.9n$ then $\lambda = \Omega\left(\sqrt{d}\right)$ as n→∞

Proof: Using relation between eigenvalue, trace of matrix and the fact that $\lambda_n = d$ we get

$$nd = 2e = tr(A^2)\sum_{i=1}^{n}\lambda_i{}^2$$

$$\le d^2 + (n-1)\lambda^2 \Rightarrow \quad \frac{nd}{d^2+(n-1)}$$

$$\le \lambda^2 \Rightarrow \sqrt{\frac{nd}{d^2+(n-1)}} \le \lambda$$

Hence we proved the claim of the Theorem 1.3.1.

The following graphs are very pseudo random.

1.3.2 Example

- The complete graph K_n. We know that the $spec(K_n) = (n-1)^1, (-1)^{n-1}$. Thus the second adjacency eigenvalue -1. (i. e. every pair of vertices are twins in the complete graph which are adjacent and have degree

$(n-1)$ by lemma 1.4.1 each of these pair defining an eigenvector associated to the eigenvalue $n/(n-1)$ but not all of them are linearly independent. We can choose the eigenvectors of the form $(1,0,0\ldots,-1,\ldots,0)^T$ where the j^th entry is -1 for all j from 2 to n. We get $(n-1)$such eigenvectors associated to eigenvalue $n/(n-1)$ which are linearly independent). So K_n is very pseudo-random. Nevertheless, we notice that the quotient of the degree and the size of K_n tend to one, i.e. $\frac{d(K_n)}{n} = (n-1)/n.$→1.

- A famous pseudo-random graph is the Payley P_q graph ([1], p.17) which has q vertices and is $(q-1)/2$ regular. In fact, P_q is a strongly regular graph with parameters$\left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right)$. By proposition 2.12 ([1], p20) the second adjacency eigenvalue of P_q equals $(P_q+1)/2$. So this graph shows that the bound λ (G) = $\Omega(\sqrt{d})$, by the extremal property of largest eigenvalue and smallest eigenvalue this is sharp.

We now extend it for non-regular graphs by taking account of the normalized Laplacian eigenvalues. Our proceeding will be analogue to the regular case.

1.3.3 Definition (**Second Laplacian Eigenvalue):** The second Laplacian eigenvalue is defined as

$$\lambda(\mathcal{L}(G)) = \max_{i\neq n} |1-\lambda_i(\mathcal{L}(G))|$$

$$= \max\{\lambda_n(\mathcal{L}(G)) - 1,\ 1-\lambda_2(\mathcal{L}(G))\}$$

If all the normalized Laplacian eigenvalues except $\lambda_1(\mathcal{L}(G)) = 0$, are near 1, i.e., The second Laplacian eigenvalue is small, then we will see that the graph is a good pseudo-random graph. First, we note that the second Laplacian eigenvalue of a regular graph is the same as the second adjacency eigenvalue of some factor.

1.3.4 Lemma: Let G be a d-regular graph. Then the second adjacency eigenvalue is d times the second Laplacian eigenvalue, i.e. $\lambda\,(A(G)) = d.\lambda(\mathcal{L}(G))$.

1.3.5 Definition: (**Volume):** The volume of a subset U of the vertices is defined as

$$\mathrm{vol}(U) = \sum_{j\in U} d_j$$

1.3.6 Theorem (**Second Laplacian Eigenvalue: Edge-Distribution ([3], p.72)):** Let G be a graph on n vertices with normalized Laplacian $\mathcal{L}(G)$ and second Laplacian eigenvalue λ. Then for any two subsets X and Y of vertices

$$\left|e(X,Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)}\right| \leq \lambda\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)}$$

There is a slightly stronger result:

1.3.7 Theorem (Second Laplacian Eigenvalue: Edge-Distribution ([3], p.73)): Let G = (V, E) be a graph on n vertices with second Laplacian eigenvalue λ Suppose X, Y are two subsets of the vertices.

Then

$$\left| e(X,Y) - \frac{vol(X)vol(Y)}{vol(G)} \right| \leq \lambda \sqrt{\frac{vol(X)vol(Y)vol(\overline{X})vol(\overline{Y})}{vol(G)}}$$

Where $\overline{X}, \overline{Y}$ denotes the complement of X, Y respectively. i.e., $\overline{X}\backslash\frac{V}{X}$.

So the second Laplacian eigenvalue of a graph G is a measure of the pseudo randomness of G. A graph has a very small second Laplacian eigenvalue than it is very pseudo-random. In the next following theorem we answer this question how small can λ be ?

MAIN RESULT II

1.3.8 Theorem: Let G be a graph on n vertices with normalized Laplacian $\mathcal{L}(G)$ and second Laplacian eigenvalue λ .Then

$$\lambda = \max_{i \neq n} |1 - (\lambda(\mathcal{L}(G))| \geq \sqrt{\frac{e_{-1}(G,G) - 1}{n-1}},$$

Where $e_{-1}(G,G) = \sum_{i=1}^{n} \sum_{k:k\sim i} \frac{1}{d_i d_k}$ in particular, if $\Delta \leq 0.9d$, then $\lambda = \Omega\left(\frac{1}{\sqrt{\Delta}}\right)$.

Proof: Using relation between eigenvalue and trace of matrix and the fact $\lambda_1(\mathcal{L}(G)) = 0$ we get

$$e_{-1}(G,G) = \sum_{i=1}^{n} \left(1 - \lambda_i(\mathcal{L}(G))\right)^2$$

$$= 1 + \sum_{i=2}^{n} \left(1 - \lambda_i(\mathcal{L}(G))\right)^2$$

$$\leq 1 + \sum_{i=2}^{n} (\lambda(\mathcal{L}(G)))^2$$

$$e_{-1}(G,G) = 1 + (n-1)\ (\lambda_i(\mathcal{L}(G)))^2$$

Solving the above inequality for λ establishes the first part of the theroem.1.3.8. We assume now $\Delta \leq 0.9d$.

Then $e_{-1}(G,G) = \sum_{i=1}^{n} \sum_{k:k\sim i} \frac{1}{d_i d_k}$

$$\geq \sum_{i=1}^{n} \sum_{k:k\sim i} \frac{1}{d_i \Delta}$$

$$= \sum_{i=1}^{n} \frac{1}{\Delta} = \frac{n}{\Delta}$$

Thus $\lambda \geq \sqrt{\frac{e_{-1}(G,G) - 1}{n-1}} \geq \sqrt{\frac{n-\Delta}{n\Delta}} \geq \sqrt{\frac{0.1}{\Delta}}$

This proves the theorem. Now we will give example that is pseudo random.

1.3.9 Example: We will now consider a regular graph and check if they are pseudo-random or not. Let K_n be the complete graph on n vertices, add k new vertices (without any edges) and then we connect each of these new vertices to all vertices of K_n. We denote the so constructed graph by G. The number of vertices of G is $n+k$. Two vertices of K_n in G are adjacent twins with degree $n+k-1$ and two vertices of the new vertices in G are non-adjacent twins. By using Faria vector we conclude that G has the normalized Laplacian eigenvalues 1 with multiplicity $k-1$ and $\frac{n+k}{n+k-1}$ with multiplicity $n-1$. Also 0 is a normalized Laplacian eigenvalue. Totally, we have found $n+k-1$ eigenvalues so far. So there is one eigenvalue left. We calculate this eigenvalue by using the fact that the sum of all normalized Laplacian eigenvalues is equal to n. Thus the last eigenvalue must be $\frac{n+2k-1}{n+k-1}$. We summarize this by spec $\mathcal{L}(G) = 0, 1^{k-1}, \frac{n+k}{n+k-1}^{n-1}, \frac{n+2k-1}{n+k-1}$

Thus G is a pseudo-random graph. Though, we observe that the maximum degree is in order of the number of edges such that we cannot apply theorem 1.3.8.

1.4. CONCLUSIONS

Throughout the paper we discuss the random and pseudo random graphs by means of eigenvalues of graph. If for a graph G the second adjacency eigenvalue $\lambda\,(A(G))$ is small then the edge distribution of G is almost the same as for the random graph G(n, d/n) by the theorem 1.2.8 graph G is pseudo-random but the theorem 1.2.9 more generalized error improved. A graph has a very small second Laplacian eigenvalue than it is very pseudo-random. The same is extending it to second normalized Laplacian eigenvalue for the measure of pseudo randomness of graph. We have answer for the question how small can be second adjacency eigenvalue of graph for measure of pseudo randomness of graph. Finally, extending it to second normalized Laplacian eigenvalue, for the measure of pseudo randomness of graphs.

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